

Erdős-Gallai-type results for colorful monochromatic connectivity of a graph*

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Abstract

A path in an edge-colored graph is called a *monochromatic path* if all the edges on the path are colored the same. An edge-coloring of G is a *monochromatic connection coloring* (MC-coloring, for short) if there is a monochromatic path joining any two vertices in G . The *monochromatic connection number*, denoted by $mc(G)$, is defined to be the maximum number of colors used in an MC-coloring of a graph G . These concepts were introduced by Caro and Yuster, and they got some nice results. In this paper, we will study two kinds of Erdős-Gallai-type problems for $mc(G)$, and completely solve them.

Keywords: monochromatic path, MC-coloring, monochromatical connection number, Erdős-Gallai-type problem.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [1]. For a graph G , we use $V(G)$, $E(G)$, $n(G)$, $m(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, number of vertices, number of edges, maximum degree and minimum degree of G , respectively. For $D \subseteq V(G)$, let $|D|$ be the number of vertices in D , and $G[D]$ be the subgraph of G induced by D .

Let G be a nontrivial connected graph with an edge-coloring $f : E(G) \rightarrow \{1, 2, \dots, \ell\}$, $\ell \in \mathbb{N}$, where adjacent edges may be colored the same. A path of G is a *monochromatic path* if all the edges on the path are colored the same. An edge-coloring of G is a

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monochromatic connection coloring (MC-coloring, for short) if there is a monochromatic path joining any two vertices in G . How colorful can an MC-coloring be? This question is the natural opposite of the recently well-studied problem on rainbow connection number [2, 4–7] for which we seek to find an edge-coloring with minimum number of colors so that there is a rainbow path joining any two vertices.

The *monochromatic connection number* of G , denoted by $mc(G)$, is defined to be the maximum number of colors used in an MC-coloring of a graph G . An MC-coloring of G is called *extremal* if it uses $mc(G)$ colors. An important property of an extremal MC-coloring is that the subgraph induced by edges with one same color forms a tree [3]. For a color i , the *color tree* T_i is the tree consisting of all the edges of G with color i . A color i is *nontrivial* if T_i has at least two edges; otherwise, i is *trivial*. A nontrivial color tree with t edges is said to *waste* $t - 1$ colors. Every connected graph G has an extremal MC-coloring such that for any two nontrivial colors i and j , the corresponding trees T_i and T_j intersect in at most one vertex [3]. Such an extremal coloring is called *simple*.

These concepts were introduced by Caro and Yuster in [3]. A straightforward lower bound for $mc(G)$ is $m(G) - n(G) + 2$. Simply color the edges of a spanning tree with one color, and each of the remaining edges may be assigned a distinct fresh color. Caro and Yuster gave some sufficient conditions for graphs attaining this lower bound.

Theorem 1 ([3]). *Let G be a connected graph with $n > 3$. If G satisfies any of the following properties, then $mc(G) = m - n + 2$.*

- (a) \overline{G} (the complement of G) is 4-connected.
- (b) G is triangle-free.
- (c) $\Delta(G) < n - \frac{2m-3(n-1)}{n-3}$. In particular, this holds if $\Delta(G) \leq (n+1)/2$, and this also holds if $\Delta(G) \leq n - 2m/n$.
- (d) $\text{Diam}(G) \geq 3$.
- (e) G has a cut vertex.

Moreover, the authors proved some nontrivial upper bounds for $mc(G)$ in terms of the chromatic number, the connectivity and the minimum degree. Recall that a graph is called *s-perfectly-connected* if it can be partitioned into $s + 1$ parts $\{v\}, V_1, \dots, V_s$, such that each V_j induces a connected subgraph, any pair V_j, V_r induces a corresponding complete bipartite graph, and v has precisely one neighbor in each V_j . Notice that such a graph has minimum degree s , and v has degree s .

- Theorem 2** ([3]). (1) *Any connected graph G satisfies $mc(G) \leq m - n + \chi(G)$.*
(2) *If G is not k -connected, then $mc(G) \leq m - n + k$. This is sharp for any k .*
(3) *If $\delta(G) = s$, then $mc(G) \leq m - n + s$, unless G is s -perfectly-connected, in which case $mc(G) = m - n + s + 1$.*

In this paper, we will study two kinds of Erdős-Gallai-type problems for $mc(G)$.

Problem A: Given two positive integers n and k with $1 \leq k \leq \binom{n}{2}$, compute the minimum integer $f(n, k)$ such that if $|E(G)| \geq f(n, k)$, then $mc(G) \geq k$.

Problem B: Given two positive integers n and k with $1 \leq k \leq \binom{n}{2}$, compute the maximum integer $g(n, k)$ such that if $|E(G)| \leq g(n, k)$, then $mc(G) \leq k$.

It is worth mentioning that the two parameters $f(n, k)$ and $g(n, k)$ are equivalent to another two parameters. Let $t(n, k) = \min\{|E(G)| : |V(G)| = n, mc(G) \geq k\}$ and $s(n, k) = \max\{|E(G)| : |V(G)| = n, mc(G) \leq k\}$. It is easy to see that $t(n, k) = g(n, k - 1) + 1$ and $s(n, k) = f(n, k + 1) - 1$. This paper is devoted to determining the exact values of $f(n, k)$ and $g(n, k)$ for all integers n and k with $1 \leq k \leq \binom{n}{2}$; see Theorem 8 and Theorem 10.

2 Main results

2.1 The result for $f(n, k)$

We first state several lemmas, which will be used to determine the value of $f(n, k)$.

Lemma 3. *Let H be a connected graph on n vertices, and G a connected spanning subgraph of H . If $mc(H) = m(H) - n + 2$, then $mc(G) = m(G) - n + 2$.*

Proof. It suffices to prove that $mc(G) \leq m(G) - n + 2$. At first, color the edges of G with $mc(G)$ colors such that there is a monochromatic path joining any two vertices. Then, give each edge in $E(H) - E(G)$ a different fresh color. Hereto we get an MC-coloring of H using $mc(G) + m(H) - m(G)$ colors, which implies that $mc(G) + m(H) - m(G) \leq mc(H)$. Therefore, $mc(G) \leq mc(H) - m(H) + m(G) = (m(H) - n + 2) - m(H) + m(G) = m(G) - n + 2$. \square

Lemma 4. *Let n and p be two integers with $0 \leq p \leq \binom{n-1}{2}$. Then every connected graph G with n vertices and $m = \binom{n}{2} - p$ edges satisfies $mc(G) \geq \binom{n}{2} - 2p$.*

Proof. Proving that $mc(G) \geq \binom{n}{2} - 2p$ amounts to finding an MC-coloring of G which wastes at most p colors. We distinguish the following two cases.

Case 1: $n - 2 \leq p \leq \binom{n-1}{2}$.

By the lower bound, we have $mc(G) \geq m - n + 2 \geq m - p = \binom{n}{2} - 2p$.

Case 2: $0 \leq p \leq n - 3$.

Now consider the graph \tilde{G} , which is obtained from \overline{G} by deleting all the isolated vertices. If $n(\tilde{G}) \leq p + 1 (\leq n - 2)$, then we can find at least two vertices v_1, v_2 of degree $n - 1$ in G . Take a star S with $E(S) = \{v_1 v : v \in \tilde{G}\}$. We give all the edges in S one color, and every other edge a different fresh color. Obviously, it is an MC-coloring of G which wastes at most p colors. If $n(\tilde{G}) \geq p + 2$, say $n(\tilde{G}) = p + t$ ($t \geq 2$), then \tilde{G} has at least t

components (since $m(\tilde{G}) = p$). If \tilde{G} has exactly two components C_1 and C_2 , then $t = 2$, $n(C_j) \geq 2$, and all the missing edges of G lie in C_j for $j \in \{1, 2\}$. Take a double star S' as follows: one vertex from C_1 is adjacent to all the vertices in C_2 , and one vertex from C_2 is adjacent to all the vertices in C_1 . Give all the edges in S' one color, and every other edge in G a different fresh color. Then we obtain an MC-coloring of G , which wastes p colors (since S' has exactly $p + 1$ edges). If \tilde{G} has $\ell \geq 3$ components C_1, C_2, \dots, C_ℓ , then $\ell \geq t$, $n(C_j) \geq 2$, and all the missing edges of G lie in C_j for $j \in \{1, 2, \dots, \ell\}$. One vertex from C_j is adjacent to every vertex in C_{j+1} by a fresh color i_j for $j \in \{1, 2, \dots, \ell\}$ (cyclically, that is a vertex from C_ℓ which is adjacent to every vertex in C_1 by the color i_ℓ). Each other edge in G receives a different fresh color. Obviously, it is an MC-coloring of G , and the number of wasted colors is $\sum_{j=1}^{\ell} (n(C_j) - 1) = p + t - \ell \leq p$. \square

As an immediate consequence, we obtain the following corollary.

Corollary 5. *Let n, p, k be three integers with $0 \leq p \leq \binom{n}{2}/2$ and $k = \binom{n}{2} - 2p$. Then $f(n, k) \leq \binom{n}{2} - p$.*

Lemma 6 ([3]). *If G is a complete r -partite graph, then $mc(G) = m - n + r$.*

Given two positive integers n and t with $3 \leq t \leq n$, let G_n^t be the graph defined as follows: partition the vertex set of the complete graph K_n into t vertex classes V_1, V_2, \dots, V_t , where $||V_j| - |V_r|| \leq 1$ for $1 \leq j \neq r \leq t$; select a vertex v_j^* from V_j ($1 \leq j \leq t$), and delete all the edges joining v_j^* to another vertex in V_j . The remaining edges in V_j ($1 \leq j \leq t$) are called *internal edges*. Clearly, $m(G_n^t) = \binom{n}{2} - n + t$. Next we will show that $mc(G_n^t) = \binom{n}{2} - 2n + 2t$. The proof is similar to that of Lemma 6. We begin with an easy observation.

Observation 1. *Let f be an extremal MC-coloring of a connected graph G . Then every nontrivial color tree in f contains at least one pair of nonadjacent vertices.*

Proof. Suppose that T_i is a nontrivial color tree, in which all the pairs of vertices are adjacent in G . Then we can adjust the coloring of T_i . Color one edge of T_i with color i , and give each other edge of T_i a different fresh color. Obviously, the new coloring is still an MC-coloring, but uses more colors than f , a contradiction. \square

Lemma 7. $mc(G_n^t) = \binom{n}{2} - 2n + 2t$.

Proof. Since G_n^t contains a spanning complete t -partite graph, it follows from Lemma 6 that $mc(G_n^t) \geq m(G_n^t) - n + t = \binom{n}{2} - 2n + 2t$. To prove the other direction, we need the following three claims.

Claim 1: In any simple extremal MC-coloring f of G_n^t , each nontrivial color tree intersects exactly two vertex classes.

Suppose that a nontrivial color tree T_i intersects $s \geq 3$ vertex classes, say V_1, V_2, \dots, V_s . Let $P_j = V(T_i) \cap V_j$ and $|P_j| = p_j$ for $1 \leq j \leq s$. Denote by x the number of internal edges in $G[\bigcup_{j=1}^s P_j]$ (the subgraph of G_n^t induced by $\bigcup_{j=1}^s P_j$). Then $G[\bigcup_{j=1}^s P_j]$ has $\sum_{1 \leq j < r \leq s} p_j p_r + x$ edges in total. Observe that T_i has $\sum_{j=1}^s p_j - 1$ edges, and since the coloring f is simple, each other edge in $G[\bigcup_{j=1}^s P_j]$ forms a trivial color tree. Thus we get that $G[\bigcup_{j=1}^s P_j]$ contains $\sum_{1 \leq j < r \leq s} p_j p_r - \sum_{j=1}^s p_j + x + 2$ colors. Now we adjust the coloring of $G[\bigcup_{j=1}^s P_j]$. One vertex from P_j is adjacent to every vertex in P_{j+1} by a fresh color i_j for $j \in \{1, 2, \dots, s\}$ (cyclically, that is a vertex from P_s which is adjacent to every vertex in P_1 by the color i_s). Each other edge in $G[\bigcup_{j=1}^s P_j]$ receives a different fresh color. Obviously, the new coloring is still an MC-coloring, but now it uses $\sum_{1 \leq j < r \leq s} p_j p_r - \sum_{j=1}^s p_j + x + s$, contradicting to the fact that f is extremal. Suppose that a nontrivial color tree T_i intersects only one vertex class, say V_1 . Clearly, $v_1^* \notin V(T_i)$, that is T_i contains no pairs of nonadjacent vertices, a contradiction. Thus each nontrivial color tree intersects exactly two vertex classes.

Claim 2: There exists a simple extremal MC-coloring of G_n^t such that each nontrivial color tree is a star or a double star, which does not contain any internal edges.

Let f be a simple extremal MC-coloring of G_n^t and T_i a nontrivial color tree in f . By Claim 1, we may assume that T_i intersects V_1 and V_2 with $1 \leq p_1 \leq p_2$. Since f is simple, any edge in $G[P_1 \cup P_2]$ but not in T_i must be a trivial color tree. Thus $G[P_1 \cup P_2]$ contains $p_1 p_2 - p_1 - p_2 + x + 2$ colors. We distinguish the following two cases (the case $p_1 = p_2 = 1$ is excluded, since then T_i has two vertices, contradicting to the fact that T_i is nontrivial).

Case 1: $p_1 = 1$ and $p_2 \geq 2$

If T_i is the star which consists of all the edges connecting P_1 and P_2 , then we are done. Otherwise, we replace T_i with this star, and color each other edge in $G[P_1 \cup P_2]$ with a different fresh color. Clearly, this change maintains an MC-coloring without affecting the total number of colors. In other words, the new coloring is still a simple extremal MC-coloring. Moreover, now the nontrivial color tree in $G[P_1 \cup P_2]$ is a star containing no internal edges.

Case 2: $2 \leq p_1 \leq p_2$.

If T_i is a double star which consists of all the edges connecting a certain vertex from P_1 and P_2 , and all the edges connecting a certain vertex from P_2 and P_1 , then we are done. Otherwise, we replace T_i with one double star as stated above, and color each other edge in $G[P_1 \cup P_2]$ with a different fresh color. Clearly, this change maintains an MC-coloring without affecting the total number of colors. In other words, the new coloring is still a simple extremal MC-coloring. Moreover, now the nontrivial color tree in $G[P_1 \cup P_2]$ is a double star containing no internal edges.

Now we may assume that every nontrivial color tree T_i in f is a star or a double star containing no internal edges. In fact, the stars can be viewed as degenerated double stars,

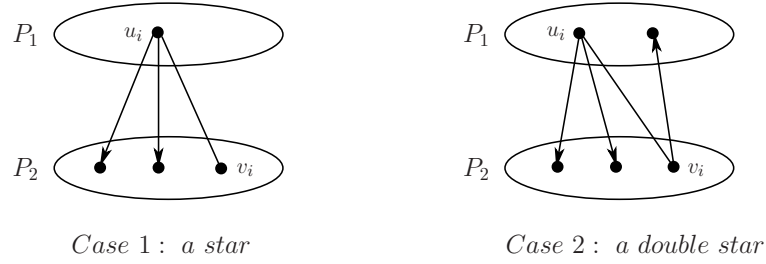


Figure 1: The illustration of Claim 2.

by letting an arbitrary leaf perform the role of the other center of a double star. So we assume that all nontrivial color trees in f are double stars (some are possibly degenerated). For a nontrivial color tree T_i , let u_i and v_i denote the two centers. Orient all the edges of T_i incident with u_i other than $u_i v_i$ (if there are any) as going from u_i toward the leaves. Similarly, orient all the edges of T_i incident with v_i other than $u_i v_i$ (if there are any) as going from v_i toward the leaves. Keep $u_i v_i$ as unoriented. Since T_i contains no internal edges, all of the edges oriented from u_i (if there are any) point to the same vertex class (the vertex class of v_i), and all of the edges oriented from v_i (if there are any) point to the same vertex class (the vertex class of u_i). Observe that the number of wasted colors of T_i is equal to the number of oriented edges in T_i .

Claim 3: For each j ($1 \leq j \leq t$), the number of edges entering V_j is at least $|V_j| - 1$.

In order to solve the monochromatic connectedness of $|V_j| - 1$ pairs of nonadjacent vertices in V_j , there are double stars T_1, T_2, \dots, T_ℓ (some are possibly degenerated). Let e_i ($1 \leq i \leq \ell$) denote the number of edges entering V_j in T_i . From Observation 1, it follows that T_i ($1 \leq i \leq \ell$) must contain the vertex v_j^* . So T_i ($2 \leq i \leq \ell$) covers at most e_i vertices in V_j but not in $\bigcup_{q=1}^{i-1} T_q$. Thus we have $(e_1 + 1) + \sum_{i=2}^{\ell} e_i \geq |V_j|$, that is, $\sum_{i=1}^{\ell} e_i \geq |V_j| - 1$.

Note that the total number of wasted colors in f is equal to the number of oriented edges in G_n^t . It follows from Claim 3 that this number is at least $\sum_{j=1}^t (|V_j| - 1) = n - t$. So we have $mc(G) \leq \left(\binom{n}{2} - n + t\right) - (n - t) = \binom{n}{2} - 2n + 2t$. \square

We are now in the position to give the exact value of $f(n, k)$.

Theorem 8. Given two positive integers n and k with $1 \leq k \leq \binom{n}{2}$,

$$f(n, k) = \begin{cases} n + k - 2 & \text{if } 1 \leq k \leq \binom{n}{2} - 2n + 4 \quad (1) \\ \binom{n}{2} + \left\lceil \frac{k - \binom{n}{2}}{2} \right\rceil & \text{if } \binom{n}{2} - 2n + 5 \leq k \leq \binom{n}{2} \quad (2) \end{cases}$$

Proof. Let G be a connected graph with n vertices and m edges. Clearly, $f(n, 1) = n - 1$, so the assertion holds for $k = 1$. If $2 \leq k \leq \binom{n}{2} - 2n + 4$, by the lower bound we know that if $m \geq n + k - 2$, then $mc(G) \geq k$, which implies $f(n, k) \leq n + k - 2$. To prove $f(n, k) \geq n + k - 2$, it suffices to find a connected graph G satisfying $m = n + k - 3$ and

$mc(G) \leq k - 1$. Let H denote the graph obtained from a copy of K_{n-2} by adding two vertices u, v and joining u to some vertices in K_{n-2} and joining v to all the other vertices in K_{n-2} . Obviously, $m(H) = \binom{n}{2} - n + 1$ and $diam(H) = 3$. Applying Theorem 1, we have $mc(H) = \binom{n}{2} - 2n + 3$. In fact, H is just the graph we want for $k = \binom{n}{2} - 2n + 4$. For $2 \leq k \leq \binom{n}{2} - 2n + 3$, we take a proper connected spanning subgraph G of H with $n + k - 3$ edges. It follows from Lemma 3 that $mc(G) = k - 1$. This completes the proof of (1).

Proving (2) amounts to showing that if $k = \binom{n}{2} - 2n + 2t + 1$ or $k = \binom{n}{2} - 2n + 2t + 2$ ($2 \leq t \leq n - 1$), then $f(n, k) = \binom{n}{2} - n + t + 1$. Let $k_1 = \binom{n}{2} - 2n + 2t + 1$, and $k_2 = \binom{n}{2} - 2n + 2t + 2$. It follows from Corollary 5 that $f(n, k_2) \leq \binom{n}{2} - n + t + 1$. Since $f(n, k_1) \leq f(n, k_2)$, if we prove $f(n, k_1) \geq \binom{n}{2} - n + t + 1$, then $f(n, k_1) = f(n, k_2) = \binom{n}{2} - n + t + 1$. So it suffices to find a connected graph G satisfying $m(G) = \binom{n}{2} - n + t$ and $mc(G) \leq k_1 - 1 = \binom{n}{2} - 2n + 2t$ for all $2 \leq t \leq n - 1$. If $t = 2$ (thus $n \geq 3$), then we can take $G = P_3, C_4$ for $n = 3, 4$, respectively; for $n \geq 5$, we take the graph G obtained from a copy of K_{n-2} by adding two adjacent vertices u, v and joining u to exactly one vertex in K_{n-2} and joining v to all the other vertices in K_{n-2} . It is easy to see that $m(G) = \binom{n}{2} - n + 2$, $\delta(G) = 2$ and u is the only vertex of degree 2. Since G is not 2-perfectly-connected, it follows from Theorem 2 that $mc(G) \leq \binom{n}{2} - 2n + 4$. If $3 \leq t \leq n - 1$, then by Lemma 7 we can take the graph G_n^t . \square

2.2 The result for $g(n, k)$

We start with a useful lemma.

Lemma 9. *Let G be a connected graph with n vertices and m edges. If $\binom{n-t}{2} + t(n-t) \leq m \leq \binom{n-t}{2} + t(n-t) + (t-2)$ for $2 \leq t \leq n - 1$, then $mc(G) \leq m - t + 1$. Moreover, the bound is sharp.*

Proof. Let f be a simple extremal MC-coloring of G . Suppose that f contains ℓ nontrivial color trees T_1, \dots, T_ℓ , where $t_i = |V(T_i)|$. Since $2 \leq t \leq n - 1$, we have $m \leq \binom{n}{2} - 1$, i.e., G is not a complete graph. Thus $\ell \geq 1$. As T_i has $t_i - 1$ edges, it wastes $t_i - 2$ colors. So it suffices to prove that $\sum_{i=1}^{\ell} (t_i - 2) \geq t - 1$. Since each T_i can monochromatically connect at most $\binom{t_i-1}{2}$ pairs of nonadjacent vertices in G , we have

$$\sum_{i=1}^{\ell} \binom{t_i-1}{2} \geq \binom{n}{2} - m.$$

Assume that $\sum_{i=1}^{\ell} (t_i - 2) < t - 1$, namely, $\sum_{i=1}^{\ell} (t_i - 1) < t - 1 + \ell$. As each T_i is nontrivial, we have $t_i - 1 \geq 2$, thus $1 \leq \ell \leq t - 2$. By straightforward convexity, the expression $\sum_{i=1}^{\ell} \binom{t_i-1}{2}$, subject to $t_i - 1 \geq 2$, is maximized when $\ell - 1$ of the t_i 's are equal

to 3, and one of the t'_i s, say t_ℓ , is as large as it can be, namely, $t_\ell - 1$ is the largest integer smaller than $t - 1 + \ell - 2(\ell - 1) = t - \ell + 1$. Hence $t_\ell - 1 = t - \ell$. Now

$$\begin{aligned} \sum_{i=1}^{\ell} \binom{t_i - 1}{2} &\leq (\ell - 1) + \binom{t - \ell}{2} \\ &= \frac{1}{2} (t^2 - t - 2 + \ell^2 + (3 - 2t)\ell) \\ &\leq \binom{t - 1}{2} \quad (\text{take } \ell = 1) \\ &< \binom{t - 1}{2} + 1. \end{aligned}$$

For a contradiction, we just need to show that $\binom{t-1}{2} + 1 \leq \binom{n}{2} - m$. In fact,

$$\begin{aligned} \binom{t-1}{2} + 1 + m &\leq \binom{t-1}{2} + 1 + \binom{n-t}{2} + t(n-t) + (t-2) \\ &= \binom{n}{2}. \end{aligned}$$

Next we will show that the bound is sharp. Let G be the graph defined as follows: at first, take a complete $(n - t + 1)$ -partite graph K with vertex classes V_1, \dots, V_{n-t+1} such that $|V_j| = 1$ for $1 \leq j \leq n - t$, and $|V_{n-t+1}| = t$; then, add the remaining edges (at most $t - 2$) to V_{n-t+1} randomly. Now assign the edges between V_1 and V_{n-t+1} with one color, and every other edge a distinct fresh color. It is easily checked that this is an MC -coloring of G using $m - t + 1$ colors, which implies $mc(G) \geq m - t + 1$. Hence $mc(G) = m - t + 1$. \square

With the aid of Lemma 9, we determine the exact value of $g(n, k)$.

Theorem 10. *Given two positive integers n and k with $1 \leq k \leq \binom{n}{2}$,*

$$g(n, k) = \begin{cases} \binom{n}{2} & \text{if } k = \binom{n}{2} \\ k + t - 1 & \text{if } \binom{n-t}{2} + t(n-t-1) + 1 \leq k \leq \binom{n-t}{2} + t(n-t) - 1 \\ k + t - 2 & \text{if } k = \binom{n-t}{2} + t(n-t) \end{cases}$$

for $2 \leq t \leq n - 1$.

Proof. If $k = \binom{n}{2}$, then clearly $g(n, k) = \binom{n}{2}$. If $\binom{n-t}{2} + t(n-t-1) + 1 \leq k \leq \binom{n-t}{2} + t(n-t) - 1$ for $2 \leq t \leq n - 1$, it follows from Lemma 9 that if $m(G) \leq k + t - 1$, then $mc(G) \leq k$. Hence, $g(n, k) \geq k + t - 1$. Now let G' be the graph as described in Lemma 9 with $k + t$ edges. Then $mc(G') = k + 1 > k$ for $\binom{n-t}{2} + t(n-t-1) + 1 \leq k \leq \binom{n-t}{2} + t(n-t) - 2$, and $mc(G') = k + 2 > k$ for $k = \binom{n-t}{2} + t(n-t) - 1$. So we have $g(n, k) \leq k + t - 1$, and

thus $g(n, k) = k + t - 1$. If $k = \binom{n-t}{2} + t(n-t)$ for $2 \leq t \leq n-1$, it follows from Lemma 9 that if $m(G) \leq k + t - 2$, then $mc(G) \leq k - 1 < k$. Hence, $g(n, k) \geq k + t - 2$. Now let G'' be the graph as described in Lemma 9 with $k + t - 1$ edges. Then $mc(G'') = k + 1 > k$. So we have $g(n, k) \leq k + t - 2$, and thus $g(n, k) = k + t - 2$. \square

References

- [1] J.A. Bondy, U.S.R. Murty, *Graph Theory*, GTM 244, Springer, 2008.
- [2] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, R. Yuster, *On rainbow connection*, Electron. J. Combin. 15(1)(2008), R57.
- [3] Y. Caro, R. Yuster, *Colorful monochromatic connectivity*, Discrete Math. 311(2011), 1786-1792.
- [4] G. Chartrand, G. Johns, K. McKeon, P. Zhang, *Rainbow connection in graphs*, Math. Bohem. 133(2008), 85-98.
- [5] M. Krivelevich, R. Yuster, *The rainbow connection of a graph is (at most) reciprocal to its minimum degree*, J. Graph Theory 63(3)(2010), 185-191.
- [6] X. Li, Y. Shi, Y. Sun, *Rainbow connections of graphs: A survey*, Graphs & Combin. 29(2013), 1-38.
- [7] X. Li, Y. Sun, *Rainbow Connections of Graphs*, SpringerBriefs in Math., Springer, New York, 2012.